AF-3555

M.A./M.Sc. (Previous)
Term End Examination, 2017-18

MATHEMATICS

Paper - I

Advanced Abstract Algebra

Time : Three Hours] [*Maximum Marks* : 100
[*Minimum Pass Marks* : 36

Note : Answer any **five** questions. All questions carry equal marks.

1. (*a*) If G is a group and H is a subgroup of index 2 in G, then show that H is a normal subgroup of G.

(*b*) Show that the set Aut (G) of all automorphism of a group G is a group under composition of mappings, and In (G) Δ Aut (G). Moreover,

$$\frac{G}{z(G)} \cong \text{In}(G)$$

2. (*a*) Prove that a group of prime order must always have a non-trivial centre.

(*b*) Define normalizer of an element. Show that normalizer of an element is a subgroup of G.

3. (*a*) If $G = \frac{Z}{(18)}$, then show that

$$(0) < \{0, 9\} < \{0, 3, 6, 9, 12, 15\}$$

$$< \{0, 1, 2, \ldots, 17\} = \frac{Z}{(18)}$$

is a composition series for $\frac{Z}{(18)}$.

(*b*) Show that if G is a commutative group having a composition series, then G is finite.

4. (*a*) Define maximal ideals. Prove that an ideal S in the ring of integers I is maximal if and only if S can be generated by an integer.

(*b*) State Zorn's lemma. If R is a commutative ring, then show that the sum of all nil ideals $A_1, A_2, \ldots, A_n$ is a nil ideal.

5. (*a*) If (N_i), $1 \le i \le k$, is a family of R-submodules of a module M, then show that

$$\sum_{i=1}^{k} N_i = \{x_1 + \ldots + x_k \mid x_i \in N_i\}$$

(*b*) Let R be a ring with unity. Let $\mathrm{Hom}_R(R, R)$ denote the ring of endomorphisms of R regarded as a right R-module. Then show that $R \cong \mathrm{Hom}_R(R, R)$ as rings.

6. (*a*) Let V be a non-zero finitely generated vector space over a field F. Then show that V admits a finite basis.

(*b*) Show that, Let R be a ring with unity, and let M be an R-module, then the following statements are equivalent

(*i*) M is simple

(*ii*) $M \neq (0)$, and M is generated by any $0 \neq x \in M$

(*iii*) $M \cong \dfrac{R}{I}$, where I is a maximal left ideal of R.

7. (*a*) Let $F \subseteq E \subseteq K$ be fields. If K is a finite extension of E and E is a finite extension of F, then show that K is a finite extension of F and $[K : F]=[K : E]\,[E : F]$.

(*b*) Show that every finite extension of a field is an algebraic extension.

8. (*a*) Prove that every field F has an algebraic closure $\overline{F}$.

(*b*) Let E be a finite extension of F. Then prove that E is a normal extension of F if and only if E is a splitting field of some polynomial over F.

9. (*a*) Prove that any two finite fields having the same number of elements are isomorphic.

(*b*) Prove that every finite separable extension of a field is necessarily a simple extension.

10. (*a*) State and prove primary decomposition theorem.

(*b*) State and prove Wedderburn-Artin theorem.